A Probabilistic Approach to Determining the Number of Widgets to Build in a Yield-Constrained Process

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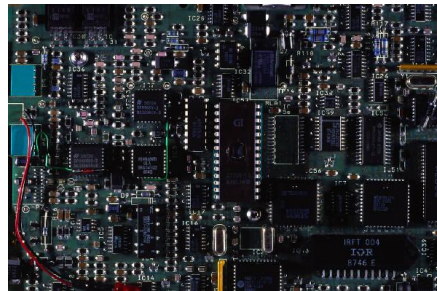
8 September 2011

Introduction

- Some cost estimating problems involve determining the number of Widgets to build in a yield-constrained manufacturing process when:
  - *It takes, on average, n attempts to produce m successes (m ≤ n)*
  - *Examples include computer chips, circuit boards, electronic components, etc.*

Computer Chips
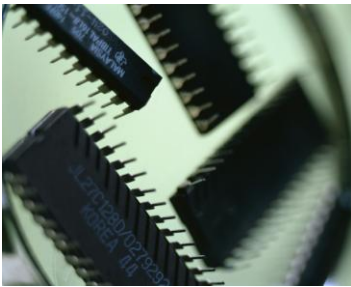
Circuit Boards

Electronics

"Failures!"

Introductory Example

- Suppose we need **four** Widgets for some military acquisition program

- Also suppose the manufacturing process is such that it takes, on average, 10 attempts to produce one working Widget

- How many "builds" will be needed to achieve our four working Widgets?

- Simplistic approach:
  - *Multiply number of Widgets needed by expected number of attempts*

    Average is 10 trials for each "success"

    So, 4 × 10 = 40 "builds"
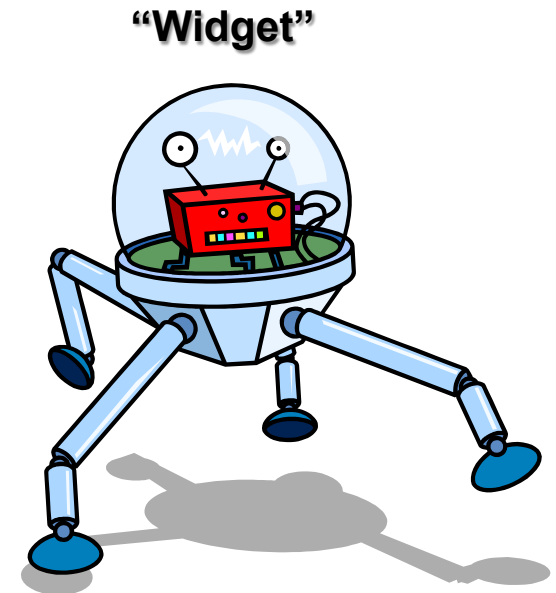
"Widget"

Introductory Example (cont'd)

- If the cost to build an individual Widget is $10,000 (whether it is a success or a failure), then the total cost would be:

$$40 \times \$10,000 = \$400,000$$

- And, we would likely still be disappointed because the probability that we would end up with four working Widgets after 40 attempts is only 57%!

  - *Consequently, there is a 43% chance that 40 Widgets is not enough!*

- Worse yet, if we want, say, 80% confidence that we will get our four working Widgets, we need to plan to build 54 of them!

  - *At a cost of 54 $10,000 = $540,000*

- Purpose of this paper:

  - *Describe the nature of the problem*

  - *Model the problem using the Negative Binomial distribution*

  - *Develop the necessary thought process to tackle the problem*

"Widget"

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Outline

- Example
- Discuss the Negative Binomial random variable and its probability distribution
- Use the Negative Binomial distribution to determine the appropriate number of Widgets to build using:
  - *The "Most Likely" approach*
  - *The "Expected Value" approach*
  - *The "Level of Confidence" approach*
- Model the number of Widgets to build as a random variable in a cost risk analysis
- Use the method to determine the MPC in a source selection

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Example

- Here is an example that was experienced some time in the past by the author while supporting an independent cost estimate[1]

  - *Space-borne application requiring four identical Widgets*
  - *Unit cost of the Widget was estimated at $500K, whether it meets specs or not*
    - Unit cost includes fabrication as well as testing to see if it meets specs (i.e., it works)
    - Those that do not meet specs are scrapped
  - *Little to no cost improvement (assume 100% learning curve)*
  - *It was known that there were yield issues in the Widget manufacturing process such that, on average, 10 Widgets had to be manufactured for every deliverable, fully functioning Widget*
  - *The question posed to the author at the time was:* **"How many Widget builds should be included in the ICE to ensure that a total of four Widgets could be delivered?"**

[1]Names, quantities, costs, and yield rate have been altered to protect the innocent!

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The Simplistic Approach

- The obvious, though simplistic, approach was to use the "expected value" technique

  – *Multiply the number of Widgets needed by the average number required to achieve success*

  – *We need four Widgets, and it takes on average 10 attempts to produce one functional Widget*

  – *Thus…*

  4 × 10 = 40 builds

  At a unit cost of $500K, the estimate is 40 × $500K = $20M

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Problems With The Simplistic Approach

- Looking at the $20M estimate from a cost-risk perspective, it is apparent that the "expected value" technique carries a substantial amount of uncertainty

- Assumes that it will actually require 40 attempts to achieve our four successes!

  - *But might we not get lucky and achieve our 4th success in fewer than 40 attempts?*

  - *Or might we fall upon hard times, in which 40 attempts will still not be enough!*
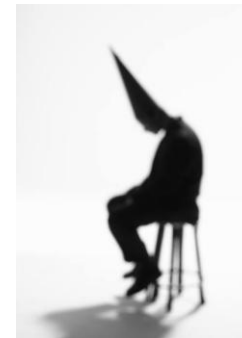
Is There Another Way to Look at This?

- Looked at ways to model this situation probabilistically
- What are the probabilities associated with our decision of how many Widgets to build?
  - *What is the probability that we will achieve our 4th success on the 40th attempt?*
  - *What is the probability that we will achieve our 4th success in fewer than 40 attempts? Fewer than 30 attempts? Etc.*
  - *What is the probability that even 40 attempts will be insufficient?*
- The Negative Binomial distribution is custom-made for questions like this!
- Preview: It turns out that the probability of achieving the 4th success on the 40th attempt is vanishingly small

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The Negative Binomial Distribution

- The Negative Binomial distribution is a discrete probability distribution that models the number of trials, *n*, needed to achieve a specified number of successes, *m* ($m \leq n$), when:
  - *The result of each trial is classified as either a success, or a failure*
  - *The probability, p, of success is the same in every trial*
  - *The trials are independent – the outcome of one trial has no influence on later outcomes*
  - *It is assumed that the mth success will occur on the nth trial*

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The Negative Binomial Random Variable

- A random variable Y_r is said to have a *negative binomial distribution* based on a series of trials with success probability p if and only if:

$$P(Y_r = y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}, \quad y = r, r+1, r+2, \ldots \text{ and } 0 \le p \le 1$$

  - *In other words, this computes the probability that the rth success will occur on the yth trial*

- The mean, variance, and cumulative distribution function of the negative binomial distribution are shown below

$$\mu = E(Y_r) = \frac{r}{p}$$

$$\sigma^2 = Var(Y_r) = \frac{r(1-p)}{p^2}$$

$$P(Y_r \le y) = \sum_{i=r}^{y} \binom{i-1}{r-1} p^r (1-p)^{i-r}$$

Answers the question: What is the probability of achieving the rth success on the yth trial?

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Calculation Examples

- Using the previous Widget example where $p = 1/10 = 0.1$, the expected number of attempts needed to achieve four successes is given as

$$\mu = E(X_4) = \frac{4}{0.1} = 40$$

- What is the probability of achieving four functional Widgets in exactly 40 attempts?

$$P(X_4 = 40) = \binom{39}{3}(0.1)^4(0.9)^{36} = 0.0206 \impliedby$$

This is the probability that the 4th success occurs on trial number 40

- Exactly 10 attempts?

$$P(X_4 = 10) = \binom{9}{3}(0.1)^4(0.9)^6 = 0.0045 \impliedby$$

This is the probability that the 4th success occurs on trial number 10

- Exactly 4 attempts?

$$P(X_4 = 4) = \binom{3}{3}(0.1)^4(0.9)^0 = 0.0001 \impliedby$$

This is the probability that the 4th success occurs on trial number 4

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Calculation Examples (cont'd)

- In reality, the manufacturing process would stop once the 4th success is achieved – which could happen on any trial

- So, a more relevant question is "What is the probability that we will achieve success in *n* or *fewer* attempts?

- What is the probability that we will achieve four functional Widgets in no more than 40 attempts?

$$P\left(X_4 \leq 40\right) = \sum_{i=4}^{40} \binom{i-1}{3} \left(0.1\right)^4 \left(0.9\right)^{i-4} = 0.5769$$

- No more than 10 attempts?

$$P\left(X_4 \leq 10\right) = \sum_{i=4}^{10} \binom{i-1}{3} \left(0.1\right)^4 \left(0.9\right)^{i-4} = 0.0128$$

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Results of Example Calculations

- Now we see that if we plan for 40 attempts, then there is nearly a 58% chance that we will achieve our four successes somewhere within those 40 attempts
  - *Unfortunately, we just don't know on which specific attempt we will be able to stop!*
- There is a 58% chance that we will have four successes in fewer than 40 attempts
- At the same time, there is a 42% chance that 40 attempts will not be enough!

- What is a cost analyst to do?!

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Determining the Right Number to Build

- The real question is: *"On which trial will we achieve the rth success?"*

- But, as we have just seen, the answer is elusive
  - *The number of the trial on which the rth success will occur is a random variable, and cannot be known with certainty*

- However, there are a few good guesses that we can make!
  - *The "Most Likely" trial number*
  - *The "Average" trial number*
 - *The trial number that corresponds to a "given level of confidence"*

The "Most Likely" Trial Number

- Consider the PMF of the negative binomial distribution for our Widget example, shown below
  - *This is the PMF for values of Y ranging from 4 to 100*
- Notice that it is maximized at a value near 30
  - *Which is less than the expected value of 40!*
- It can be shown that the negative binomial distribution is maximized at:

$$y_{\text{mode}} = \left\lfloor 1 + \frac{r-1}{p} \right\rfloor$$

- For this example, that equates to

$$y_{\text{mode}} = \left\lfloor 1 + \frac{4-1}{0.1} \right\rfloor = 31$$
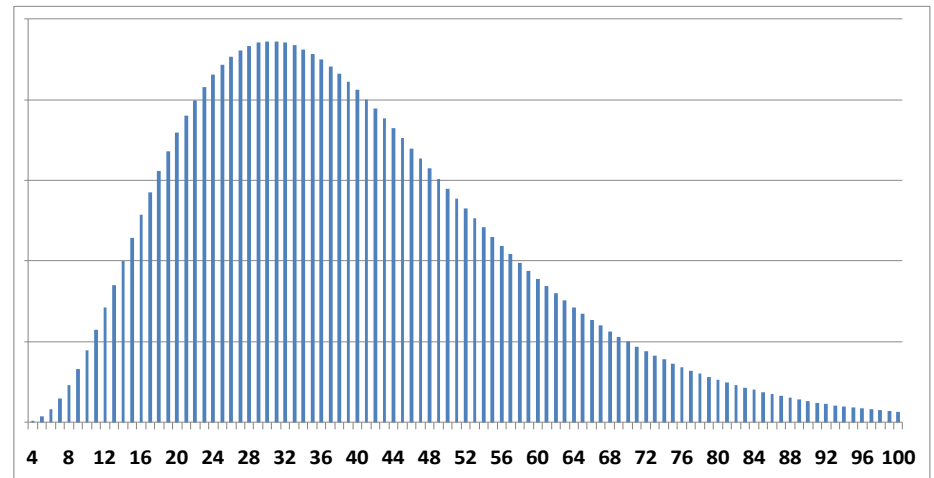
The "Most Likely" Trial Number (cont'd)

- Using the "Most Likely" trial number approach, we can say that the *most likely* trial number on which the 4th success will occur is the 31st trial
  - *So we estimate the cost of building 31 Widgets*
- The drawback is that there is still a very low probability that it will take exactly 31 attempts to achieve four successful Widgets, and the probability of achieving four successes in 31 *or fewer* attempts is also significantly small

$$P\left(X_4 = 31\right) = \binom{30}{3}\left(0.1\right)^4\left(0.9\right)^{27} = 0.0236 \qquad P\left(X_4 \leq 31\right) = \sum_{i=4}^{31}\binom{i-1}{3}\left(0.1\right)^4\left(0.9\right)^{i-4} = 0.3762$$

- But there is nearly a 38% chance that we will be successful within 31 trials, so it may be worth the risk
  - *In this case our cost estimate would be $500K × 31 = $15.5M*

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The "Average" Trial Number

- One might choose to use this approach in order to have more confidence that a sufficient number of trials are planned
- We have seen this before – it is the "simplistic method" described previously
- The "average" trial number is simply the mean, or expected value, of the negative binomial distribution

$$\mu = E\left(X_r\right) = \frac{r}{p}$$

- For our Widget example, this equates to 40 trials

$$\mu = E\left(X_4\right) = \frac{4}{0.1} = 40$$

The "Average" Trial Number (cont'd)

- Using the "Average" trial number approach, we can say that the *expected,* or *average* trial number on which the 4th success will occur is the 40th trial
  - *So we estimate the cost of building 40 Widgets*
- As before, though, there is still a very low probability that it will take exactly 40 attempts to achieve four successful Widgets, and the probability of achieving four successes in 40 *or fewer* attempts is also significantly small

$$P\left(X_4 = 40\right) = \binom{39}{3}\left(0.1\right)^4\left(0.9\right)^{36} = 0.0206 \qquad P\left(X_4 \le 40\right) = \sum_{i=4}^{40}\binom{i-1}{3}\left(0.1\right)^4\left(0.9\right)^{i-4} = 0.5769$$

- But there is nearly a 58% chance that we will be successful within 40 trials, so the added confidence may be worth the cost
  - *In this case our cost estimate would be $500K × 40 = $20.0M*

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The "Level of Confidence" Approach

- Using this approach, we specify a level of confidence in advance, then determine the number of trials needed to achieve the rth success based on that confidence level

  - *Examples are: 30%, 50%, 80%, etc.*

- Referring to our Widget example, suppose management is comfortable with an 80% confidence level that the number of Widgets built will deliver four successes

- A plot of the CDF can be used to determine the necessary number of trials

- In this case, the 80th percentile is 54 trials, so 54 trials will give us the desired level of confidence

  - *And, in this case our cost estimate would be $500K   54 = $27.0M*
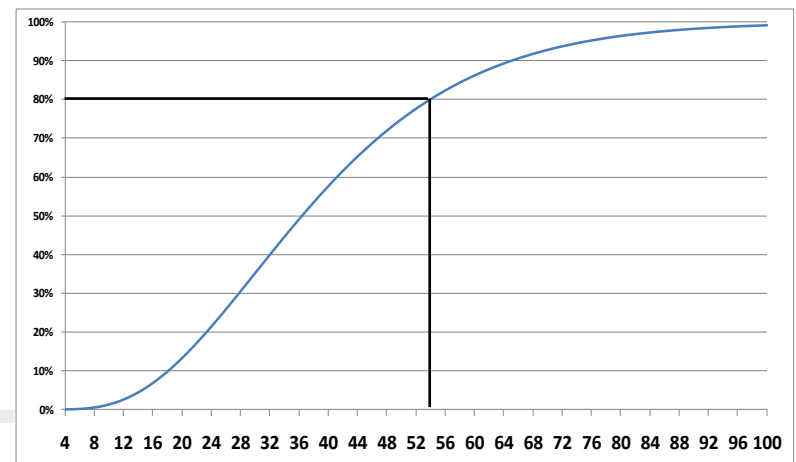
The "Level of Confidence" Approach (cont'd)

- Another way to view this is develop a table of probabilities such as that shown below:

Avg. Number of attempts needed, n: **10** to achieve one success
Number of Widgets needed, m: **4**

Expected number of trials needed: **40**
Std Dev: **19.0**

| Trial no., Y	P(Y = y)	P(Y <= y)	Trial no., Y	P(Y = y)	P(Y <= y)	Trial no., Y	P(Y = y)	P(Y <= y)	Trial no., Y	P(Y = y)	P(Y <= y)
4	0.0001	0.0001	21	0.0190	0.1520	38	0.0216	0.5352	55	0.0115	0.8130
5	0.0004	0.0005	22	0.0200	0.1719	39	0.0211	0.5563	56	0.0110	0.8240
6	0.0008	0.0013	23	0.0208	0.1927	40	0.0206	0.5769	57	0.0104	0.8344
7	0.0015	0.0027	24	0.0215	0.2143	41	0.0200	0.5969	58	0.0099	0.8443
8	0.0023	0.0050	25	0.0221	0.2364	42	0.0195	0.6164	59	0.0094	0.8537
9	0.0033	0.0083	26	0.0226	0.2591	43	0.0189	0.6352	60	0.0089	0.8626
10	0.0045	0.0128	27	0.0230	0.2821	44	0.0182	0.6534	61	0.0084	0.8710
11	0.0057	0.0185	28	0.0233	0.3054	45	0.0176	0.6711	62	0.0080	0.8790
12	0.0071	0.0256	29	0.0235	0.3290	46	0.0170	0.6881	63	0.0076	0.8866
13	0.0085	0.0342	30	0.0236	0.3526	47	0.0164	0.7044	64	0.0071	0.8937
14	0.0100	0.0441	31	0.0236	0.3762	48	0.0157	0.7201	65	0.0067	0.9004
15	0.0114	0.0556	32	0.0235	0.3997	49	0.0151	0.7352	66	0.0064	0.9068
16	0.0129	0.0684	33	0.0234	0.4231	50	0.0145	0.7497	67	0.0060	0.9128
17	0.0142	0.0826	34	0.0231	0.4462	51	0.0139	0.7636	68	0.0056	0.9184
18	0.0156	0.0982	35	0.0228	0.4690	52	0.0133	0.7768	69	0.0053	0.9238
19	0.0168	0.1150	36	0.0225	0.4915	53	0.0127	0.7895	70	0.0050	0.9288
20	0.0180	0.1330	37	0.0221	0.5136	54	0.0121	0.8015	71	0.0047	0.9335

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Do We HAVE to Pick a Number?

- If we are developing a point estimate, then we need to be able to specify a quantity
- But, so far we have seen that it is impossible to determine, with precision, the exact number of trials needed to produce m successful widgets under these circumstances
  - *The best we can do is to make an educated guess*
- Perhaps it would be better to use our knowledge of the uncertainty in a cost risk analysis
- We model the uncertainty of CERs and cost drivers
  - *Quantity is ultimately an input variable at some point…*
  - *Why not model the quantity as a random variable also, using the Negative Binomial distribution?*

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Cost Risk Analysis Example

- Suppose that a (fictional) CER for Widgets (whether they work or not) is given as shown at right

- Suppose there is no significant learning curve, so the cost of the nth Widget is the same as the cost of the first Widget

- Suppose we know that it takes, on average, 10 attempts to build one fully functional Widget, and that we need four of them

- The cost risk approach should model:
  - *The uncertainty of the CER*
  - *The uncertainty of the input variable (area)*
  - *The uncertainty of the number that need to be built to ensure four working Widgets*

$$\text{Widget Cost (FY11\$K)} = 57.17 + 0.3 \left(Area \left(sq.in.\right)\right)^{1.82}$$

$$SPE = 22\%, \quad Pearson's \ R^2 = 84\%, \quad Bias = 0$$


Fictional Widget CER
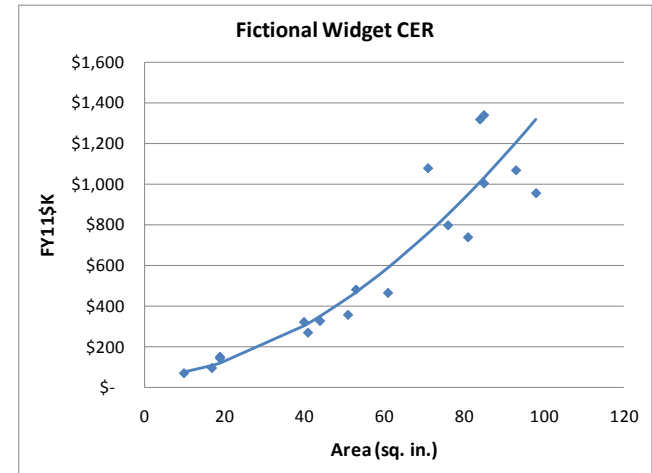
Cost Risk Analysis Example (cont'd)

- Suppose for simplicity that the area (sq. in.) of the Widgets is fixed by design, so no need to model its uncertainty

- Using @RISK, we could model the CER uncertainty as lognormal, and the build quantity uncertainty as negative binomial

- If the area of the Widgets is 55.12 sq. in., then the spreadsheet would look like the following

- The point estimate is $20M for 40 attempts at building the Widgets

- But, lets take a look at the cost probability distribution after a Monte Carlo simulation…

Cost Risk Analysis Example (cont'd)

- Here we see the combined probability distribution in which the number of attempts needed, and the cost of each attempt, are modeled as random variables
  - *P(Cost ≤ $20.0M) = 58.4%*
  - *P(Cost ≤ $27.6M) = 80.0%*
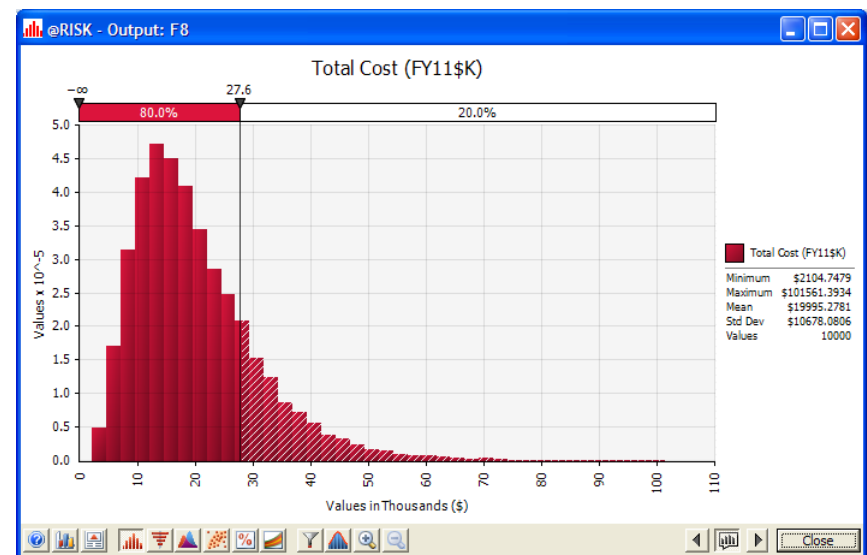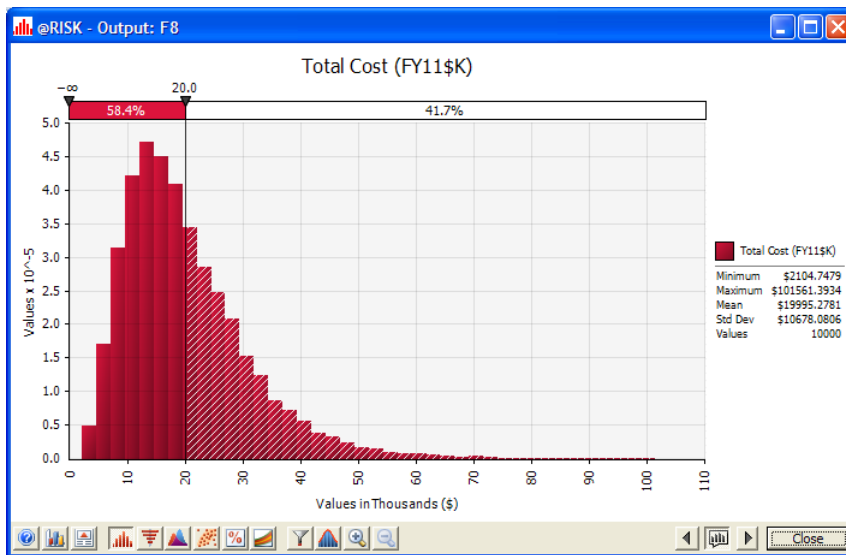
Cost Risk Analysis Example (cont'd)

- Recall that when we assumed a fixed cost of $500K per Widget, the 80% confidence level result was to build 54 Widgets at a cost of $27.0M

- The convolved cost distribution gives a consistent answer, but does not require us to worry about "how many to build"
  - *If we simply budget $27.6M, we have an 80% chance of success*
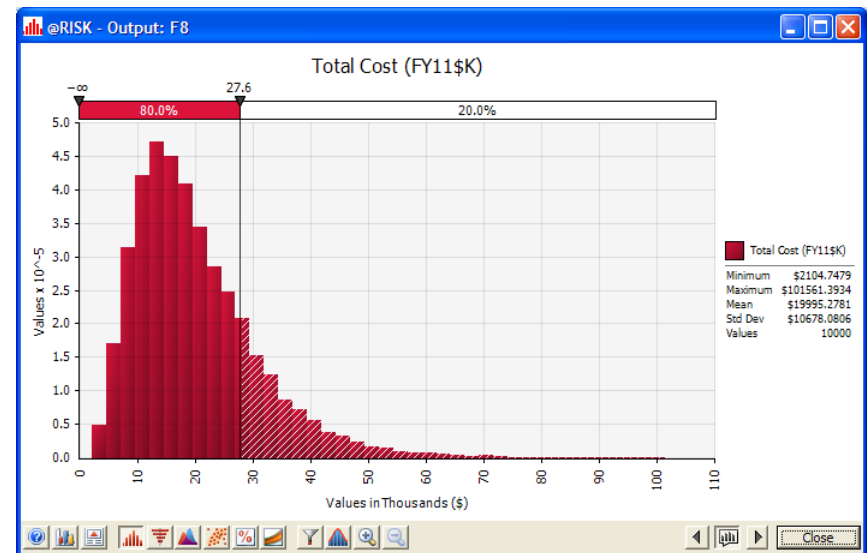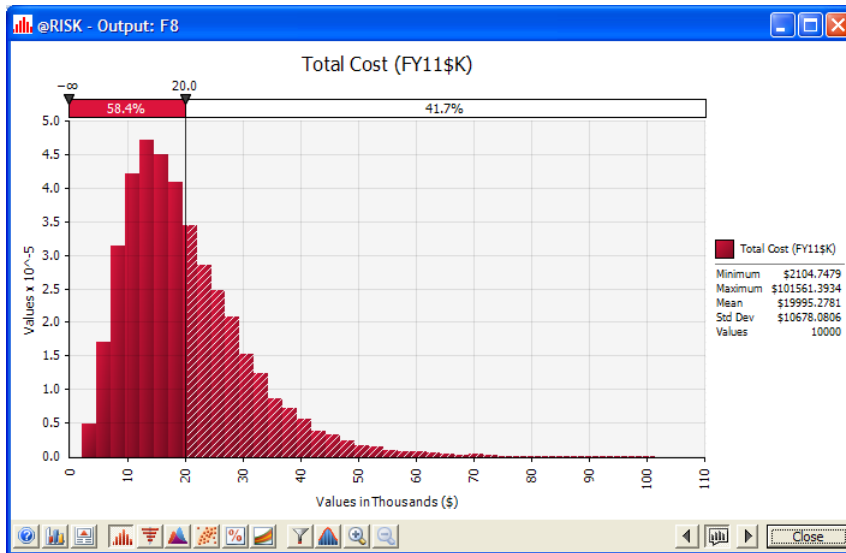
Modeling Uncertainty in a Source Selection Evaluation

- The negative binomial distribution can also be used to assist in determining the most probable cost (MPC) in a government source selection

- In a competitive proposal, an offeror's proposed cost is likely to be lower than it should be

  - *Offerors are motivated to bid the lowest price possible while still being plausibly reasonable and realistic*

  - *Future government-directed changes will enable additional opportunities to refine proposed design, make enhancements, re-establish program management baseline, and increase fee pool*

- So, in a source selection, the government computes an MPC to more accurately represent the "real" cost of the contract

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The Contractor Proposal for our Example

- Suppose an offeror bids a price in which it is assumed that four functional Widgets can be produced in a minimum amount of trials
- The offeror may bid, say, the 20th percentile quantity
  - *Defendable, yet optimisitic*
- Furthermore, the offeror may bid an optimistic unit cost for the Widgets, say $400K
  - *Even though our CER says it should cost $500K*
- This would result in 24 builds, yielding a cost proposal of 24 $400K = $9.6M for the four Widgets
- But, using the assumptions in our Example, the unit cost is probably too low, and there is only a 1 in 5 chance that the offeror would be successful with 24 builds

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The Government MPC for our Example

- The government, on the other hand, would estimate the unit cost at $500K, and, to be consistent with the term MPC, might choose to use the "Most Likely Trial Number" method to arrive at the quantity to build

$$y_{\text{mode}} = \left\lfloor 1 + \frac{r-1}{p} \right\rfloor = \left\lfloor 1 + \frac{4-1}{0.1} \right\rfloor = 31$$

- The resulting MPC for the Widgets is 31 × $500K = $15.5M
- However, the source selection team is not constrained to use the "Most Likely Trial Number," but may choose to use of the other "more probable" methods
  – *"Average Trial Number"*
  – *"Trial Number that Corresponds to a Given Level of Confidence"*

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Summary and Conclusions

- Described the nature of the problem of determining the number of Widgets to build in a yield-constrained manufacturing process
  - *Modeled the number to build as a negative binomial random variable*
- Concluded that there is no "best answer" to the question
  - *But came up with some "good guesses" using the "most likely trial number," the "average trial number," and "the trial number that corresponds to a given level of confidence" methods*
- Discussed how one could use these methods to model the number of Widgets as a random variable in a cost risk analysis, and in a source selection MPC

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Potential Future Research

- So far, we've avoided the question of what to do if the manufacturing process "improves" over time
  - *Meaning the probability of success increases with increased numbers of trials*
- Future research should include a study of how to determine the number of Widgets to build if the success probability is not constant

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Questions or Comments?

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Backups

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The Bernoulli Random Variable…

- Suppose a Widget manufacturing process is such that it takes, on average, *n* attempts (trials) in order to produce *one* satisfactory Widget (a success)

- One can then view each attempt as a Bernoulli random variable with probability of success, *p*

$$P\left(\text{success}\right) = p = \frac{1}{n}$$

provided the trials have these critical properties:

1. *The result of each trial is classified as either a success, or a failure;*
2. *The probability of success, p, is the same in every trial; and*
3. *The trials are independent – the outcome of one trial has no influence on later outcomes*

Answers the question: What is the probability of success on any given trial?

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Next, the Binomial Random Variable…

- Now we extend the concept of Bernoulli random variables to count the number of successes, *m*, in *n* repeated Bernoulli trials, each with probability of success, *p*

- A binomial experiment should have the following properties:

  1. *The experiment consists of n identical trials*

  2. *Each trial results in one of two outcomes – success, or failure*

  3. *The probability of success on a single trial is p, and remains constant from trial to trial*

  4. *The probability of failure on a single trial is (1-p) = q*

  5. *The trials are independent*

  6. *The random variable of interest is Y, the number of successes observed during the n trials*

The Binomial Random Variable (cont'd)

- A random variable Y is said to have a *binomial distribution* based on n trials with success probability p if and only if:

$$P(Y = m) = \binom{n}{m} p^m (1 - p)^{n-m}, \quad m = 0, 1, 2, \ldots, n, \text{ and } 0 \le p \le 1$$

where

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

- The mean, variance, and cumulative distribution function of the binomial distribution are shown below

$$\mu = E(Y) = np$$
$$\sigma^2 = Var(Y) = np(1-p)$$

$$P(Y \le y) = \sum_{i=1}^{y} \binom{n}{i} p^i (1 - p)^{n-i}$$

Answers the question: What is the probability of achieving m successes in n trials?

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The Binomial Random Variable (cont'd)

- Unfortunately, the binomial random variable does not quite address the problem at hand
  - *Because it counts the number of successes in n trials, when what we want is the number of trials needed to achieve m successes*
- But, it represents the next step in the buildup to the negative binomial distribution

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Then, the Geometric Random Variable

- The Geometric Random Variable is defined for an experiment that is very similar to the binomial experiment
  - *Also concerned with identical and independent trials, each of which can result in either a success or a failure*
  - *The probability of success is p, and is constant from trial to trial*
- However, instead of counting the number of successes that occur in *n* trials, the geometric random variable represents the individual trial number on which the first success occurs
- The experiment consists of a series of trials, and concludes with the *first* success

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The Geometric Random Variable (cont'd)

- A random variable *Y* is said to have a *geometric distribution* based on a series of trials with success probability *p* if and only if:

$$P(Y = y) = (1 - p)^{y-1} p, \quad y = 1, 2, 3, \ldots \text{ and } 0 \leq p \leq 1$$

  - *In other words, the probability that the first success occurs on trial number y is computed as (y-1) failures, followed by one success*

- The mean, variance, and cumulative distribution function of the geometric distribution are shown below

$$\mu = E(Y) = \frac{1}{p}$$

$$\sigma^2 = Var(Y) = \frac{1 - p}{p^2}$$

$$P(Y \leq y) = \sum_{i=1}^{y} (1 - p)^{y-1} p$$

Answers the question: What is the probability of achieving the first success on the yth trial?

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The Geometric Random Variable (cont'd)

- As with the binomial random variable, the geometric random variable does not quite address our problem
  - *Because it counts the number of trials needed to achieve the first success, when what we want is the number of trials to achieve m successes*
- However, this is our last stop on the road to defining the negative binomial distribution!